The setting

- Nonlinear dynamical system: $u(t) = \Phi_t(u(0); \theta)$.
- Ecological model (Ricker map) as motivation: $N_t = rN_{t-1}e^{-N_{t-1}+e_{t-1}}$, $e_t \sim \mathcal{N}(0, \sigma_e^2)$,
- Data observed $y_t \sim \mathrm{Pois}(\phi N_t)$ so $\theta = (r, \sigma_e^2, \phi)$. Call the full dataset y from here on in.
- Likelihood is numerically horrific. How to infer θ given data y?

The basic idea

- Log-likelihood poorly behaved — how to smooth the log-likelihood surface?
- Know from central limit theorem: if $\{x_1, x_2, \dots, x_n\}$, $\mathbb{E}[x_i] = \mu$, $\mathrm{Var}[x_i] = \sigma^2$, are draws from some probability distribution then $\sqrt{n}\,(\bar{x}_n - \mu) \to \mathcal{N}(0, \sigma^2)$. Generalizes to multivariate setting as you would expect.
- Now reduce y to s, a vector of summary statistics, and assume $\mathrm{s} \sim \mathcal{N}(\boldsymbol{\mu}_\theta, \Sigma_\theta)$.
- Given a value of θ we then generate a set of replicate datasets $\{\mathrm{y}_1^*, \mathrm{y}_2^*, \dots\}$ and corresponding summary statistics $\{\mathrm{s}_1^*, \mathrm{s}_2^*, \dots\}$. Then we use these to compute the log likelihood (to a constant)

$$l_{\mathrm{s}}(\theta) = -\frac{1}{2}\,(\mathrm{s} - \hat{\mu}_\theta)\,\hat{\Sigma}_\theta^{-1}\,(\mathrm{s} - \hat{\mu}_\theta)^\top\,,$$

where $\hat{\mu}_\theta$ and $\hat{\Sigma}_\theta$ are each estimated from the replicates s_i^*. This is a proxy for assessing the likelihood of θ through the summary statistics.

Inference for θ

Use MCMC. Given a starting point $\theta^{[0]}$ we proceed for each k

1. Generate $\theta^* \sim q(\theta^{[k-1]})$.
2. Compute $\alpha = \exp(l_s(\theta^*) - l_s(\theta^{[k-1]})$
3. Set $\theta^{[k]} = \theta^*$ w.p. α, otherwise $\theta^{[k]} = \theta^{[k-1]}$.

Samples drawn from l_s

Case study

- Adult blowfly populations are given from the delay differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = PN(t-\tau)e^{N(t-\tau)/N_0} - \delta N(t).$$

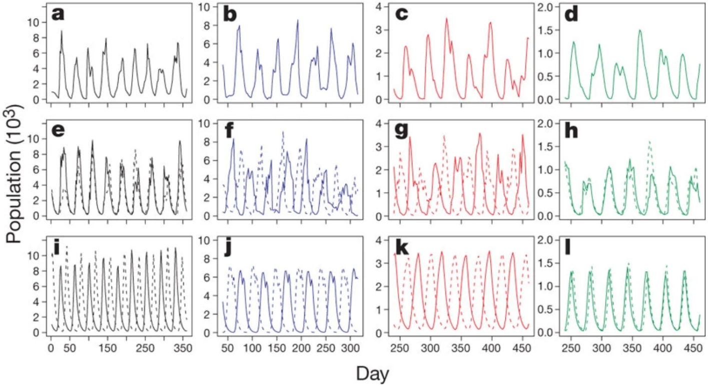
  Discretized to give

$$N_{t+1} = R_t + S_t,$$

  where $R_t \sim \mathrm{Pois}(PN_{t-\tau}\exp(-N_{t-\tau}/N_0)e_t)$, and $S_t \sim \mathrm{Binom}(\exp(-\delta\varepsilon_t), N_t)$. Stochastic terms e_t, ε_t are Gamma random variables with unit means and variances σ_p^2 and σ_d^2.

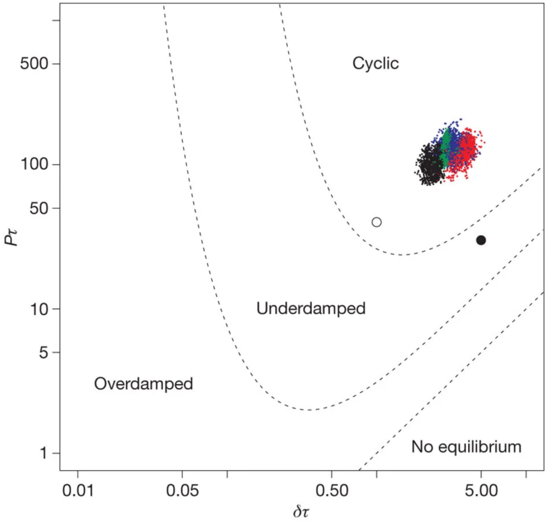
- I have no idea how these come about.
- Parameters thus $\theta = (P, N_0, \delta, \tau, \sigma_p, \sigma_d)$.

Results



a, **b**, Two laboratory adult populations of sheep blowfly maintained under adult food limitation[4,5]. **c**, **d**, As in **a** and **b** but maintained under moderate and more severe juvenile food limitation[4]. **e**–**h**, Two replicates (one solid, one dashed) from the full model (equation (4)) fitted separately to the data shown in each of panels **a**–**d**, immediately above. **i**–**l**, As in **e**–**h** for the model with demographic stochasticity only. The observations are made every second day. The simulation phase is arbitrary. Notice the qualitatively good match of the dynamics (**e**–**h**) of the full model (equation (4)) to the data, relative to the insufficiently variable dynamics of the model with demographic stochasticity only (**i**–**l**).

Results



The coloured points are samples from the stability-controlling parameter combinations $\delta\tau$ and $P\tau$, plotted (with matching colour coding) for each experimental run shown in Fig. 3. The open and filled circles show stability properties for alternative chain starting conditions: they give indistinguishable results, although the conditions marked by the filled circle lie in the plausible range for external noise-driven dynamics[14]. The dynamics comprise limit cycles perturbed by noise but not driven by noise. The fluctuations are driven by the intrinsic population-dynamic processes, not by random variation exciting a resonance in otherwise stable dynamics.

Discussion

- Why no Bayes? Would make sense to incorporate prior knowledge in the model.
- How to choose summary statistics? This is given lengthy discussion in the SI.
- What do parameter estimates/sampling results look like? This is shown in the SI but would be nice to see?