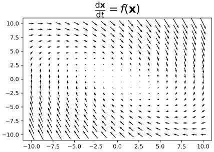
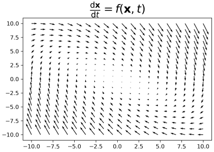
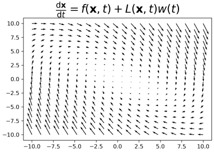
Itô Integration

January 22, 2021

The Problem







$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t) \xLeftrightarrow[\text{of Calculus}]{\text{Fundamental Theorem}} x(t) = x(0) + \int_0^t f(x,s)\mathrm{d}s$$

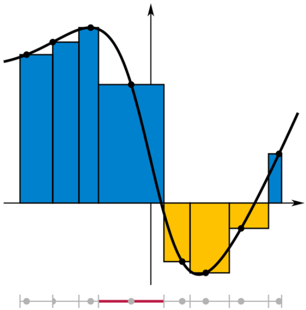
$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,t) + L(x,t)W(t) \\ x(t) = x(0) \quad &+ \int_0^t f(x,s)\mathrm{d}s \\ &+ \text{"} \int_0^t L(x,s)\mathrm{d}B_s\text{"} \end{aligned}$$

Integration Theory

Riemann Integral

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n-1} f(\xi_i)(t_{i+1} - t_i)$$

$$\xi_i \in [t_i, t_{i+1})$$



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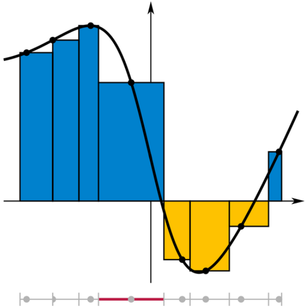
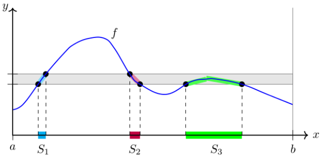


Lebesgue Integral

$$\int_a^b f(x)\mu(\mathrm{d}x)$$



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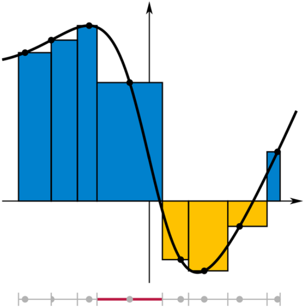

Riemann-Stieltjes Integral

$$\int_a^b f\mathrm{d}g = \lim_{n\to\infty} \sum_{i=1}^{n-1} f(\xi_i)(g(t_{i+1}) - g(t_i))$$

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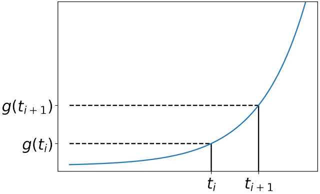
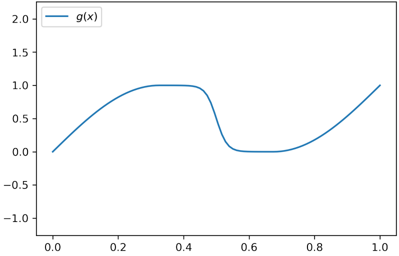


Total Variation

Given a continuous function g, it's total variation over the interval $[a, b]$ is defined as

$$V_{[a,b]}(g) = \sup_{a=t_1<\cdots<t_n=b} \sum_{i=1}^{n-1} |g(t_{i+1}) - g(t_i)|$$



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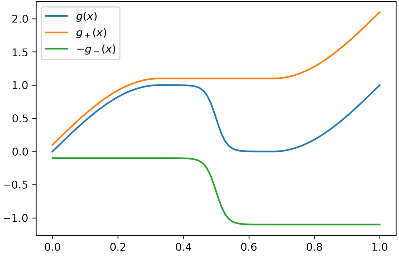

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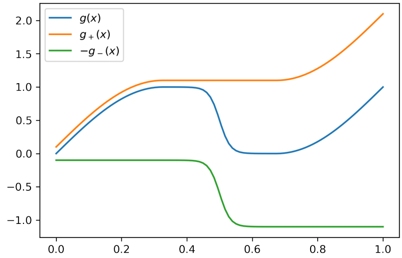

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Stieltjes integral for bounded variation functions is defined as:

$$\int_a^b f \mathrm{d}g = \int_a^b f \mathrm{d}g_+ - \int_a^b f \mathrm{d}g_-$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n-1} f(\xi_i)(g(t_{i+1}) - g(t_i))$$

Stieltjes integral interpretation

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Total profits – $\int_0^T S\mathrm{d}p$
If p is differentiable $\int_0^T S\mathrm{d}p = \int_0^T S(t)p'(t)\mathrm{d}t$.

Itô integral

We want our definition of $\int_a^b X_t \mathrm{d}B_t$ to be similar, but B_t has unbounded total variation!
We now have to make a specific choice of ξ:

1. $\xi = t_i$ gives rise to the Itô integral:

$$\int_a^b X_t \mathrm{d}B_t = \lim_{n \to \infty} \sum_{i=1}^{n-1} X_{t_i}(B_{t_{i+1}} - B_{t_i})$$

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2. $\xi = \frac{t_i + t_{i+1}}{2}$ corresponds to the Stratonovitch integral:

$$\int_a^b X_t \circ \mathrm{d}B_t = \lim_{n\to\infty} \sum_{i=1}^{n-1} X_{\frac{t_i+t_{i+1}}{2}}(B_{t_{i+1}} - B_{t_i})$$

Properties of Itô integral

- The stochastic processes that we can integrate over with the Itô integral have to satisfy:
  1. *some measurability and adaptivity conditions*
  2. $\mathbb{E}[\int_a^b X_t^2 \mathrm{d}t] < \infty$ (or more loosely $\mathbb{P}[\int_a^b X_t^2 \mathrm{d}t < 1] = 1$).

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- $\mathbb{E}[\int_a^b X_t \mathrm{d}B_t] = 0$
- **Integration by parts**: if f is deterministic, continuous and of bounded variation:

$$\int_a^b f(t)\mathrm{d}B_t = f(b)B_b - f(a)B_a - \int_a^b B_t \mathrm{d}f(t)$$

Itô's lemma

Itô's process is a stochastic process X_t such that $X_t = X_0 + \int_0^t U_s \mathrm{d}s + \int_0^t V_s \mathrm{d}B_s$ (for reasonable U_s and V_s).

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Itô's lemma: If X_t is an Itô process, and $g(t,x)$ is twice cts. differentiable, then $Y_t = g(t, X_t)$ is also an Itô process such that

$$Y_t = Y_0 + \int_0^t \frac{\partial g}{\partial s}(s, X_s)\mathrm{d}s + \int_0^t \frac{\partial g}{\partial x}(s, X_s)\mathrm{d}X_s + \int_0^t \frac{\partial^2 g}{\partial x^2}(s, X_s)(\mathrm{d}X_s)^2$$

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where

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\int_0^t Z_s \mathrm{d}X_s &= \int_0^t Z_s U_s \mathrm{d}s + \int_0^1 Z_s V_s \mathrm{d}B_s \\
\mathrm{d}X_t &= U_s \mathrm{d}s + V_s \mathrm{d}B_s \\
(\mathrm{d}X_t)^2 &= (\mathrm{d}X_t) \cdot (\mathrm{d}X_t) \\
&= (U_s \mathrm{d}s + V_s \mathrm{d}B_s) \cdot (U_s \mathrm{d}s + V_s \mathrm{d}B_s) \\
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$$\int_0^t Z_s \mathrm{d}X_s = \int_0^t Z_s U_s \mathrm{d}s + \int_0^1 Z_s V_s \mathrm{d}B_s$$

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Example

Solve $\frac{\mathrm{d}N_t}{\mathrm{d}t} = (r + \alpha W_t)N_t$ with some given N_0

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In Itô interpretation, this becomes: find N_t s.t. $N_t = N_0 + \int_0^t rN_s \mathrm{d}s + \int_0^t \alpha N_s \mathrm{d}B_s$

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In differential form: $\mathrm{d}N_t = rN_t\mathrm{d}t + \alpha N_t \mathrm{d}B_t$ or $\frac{\mathrm{d}N_t}{N_t} = r\mathrm{d}t + \alpha \mathrm{d}B_t$

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Claim: $\int_0^t \frac{\mathrm{d}N_s}{N_s} = rt + \alpha B_t$

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Proof: $\int_0^t \frac{\mathrm{d}N_s}{N_s} = \int_0^t \frac{rN_s}{N_s}\mathrm{d}s + \int_0^t \frac{\alpha N_s}{N_s}\mathrm{d}B_s = rt + \alpha B_t$

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Consider $g(t,x) = \log x$. Then for $Y_t = \log N_t$

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \frac{\mathrm{d}N_s}{N_s} + \frac{1}{2}\int_0^t -\frac{1}{N_s^2}(\mathrm{d}N_t)^2 \\ &= Y_0 + \int_0^t \frac{\mathrm{d}N_s}{N_s} - \frac{\alpha^2 t}{2} \end{aligned}$$

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So $\log \frac{N_t}{N_0} = \int_0^t \frac{\mathrm{d}N_s}{N_s} - \frac{\alpha^2 t}{2} = rt + \alpha B_t - \frac{\alpha^2 t}{2} \implies N_t = N_0 \exp(rt + \alpha B_t - \frac{\alpha^2 t}{2})$

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\end{aligned}
$$

So $\log \frac{N_t}{N_0} = \int_0^t \frac{\mathrm{d}N_s}{N_s} - \frac{\alpha^2 t}{2} = rt + \alpha B_t - \frac{\alpha^2 t}{2} \implies N_t = N_0 \exp(rt + \alpha B_t - \frac{\alpha^2 t}{2})$

Stratonovich interpretation gives the solution $\overline{N_t} = N_0 \exp(rt + \alpha B_t)$

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- The two are equivalent in the sense that

$$
\begin{aligned}
\mathrm{d}X_t &= F(t, X_t)\mathrm{d}t + L(t, X_t) \circ \mathrm{d}B_t \\
\mathrm{d}X_t &= F(t, X_t)\mathrm{d}t + L(t, X_t)\mathrm{d}B_t + \frac{1}{2}\frac{\partial^2 L}{\partial x^2}(t, X_t)L(t, X_t)\mathrm{d}t
\end{aligned}
$$

Give the same solutions.

Discussion

- Intuition behind quadratic variation.
- Connection between white noise and Itô and Stratonovich integrals. I.e. what's the precise definition of the derivative of Brownian motion? Does it coincide with either of the integrals?