Statistics of Linear Stochastic Differential Equations - CSML Reading Group

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February 13, 2021

Motivation

The linear time dependent SDE is written down as

$$d\boldsymbol{x} = \boldsymbol{F}(t)\boldsymbol{x}dt + \boldsymbol{u}(t)dt + \boldsymbol{L}(t)d\boldsymbol{\beta} \quad \boldsymbol{x}_0 \sim \mathcal{N}(\boldsymbol{m}_0, \boldsymbol{P}_0) \tag{1}$$

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$\boldsymbol{x}(t) \in \mathbb{R}^D$, $\boldsymbol{u}(t) \in \mathbb{R}^d$ is a vector valued function of time (an input to the linear system), and $\beta(t) \in \mathbb{R}^s$ is a Brownian motion with diffusion matrix $\boldsymbol{Q}$.

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Example: Statistics of the Ornstein–Uhlenbeck process

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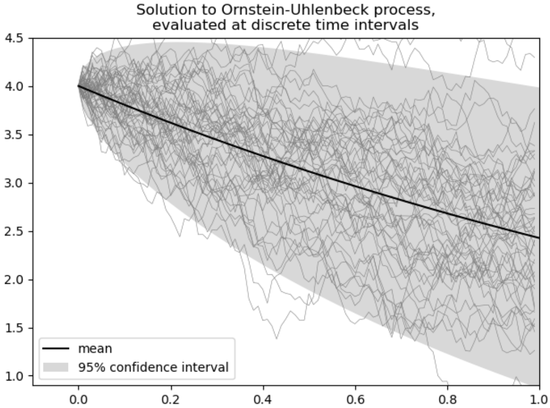

Figure: 50 realisations of the Ornstein–Uhlenbeck process, solved for discrete time. The **black line** is the mean $m(t)$ and the mean $m(t)$ and covariance $P(t)$ define the **quantiles**. How do we evaluate the mean $m(t)$ and covariance $P(t)$?

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Note that Eq. (5) is the Lyapunov differential equation.

Ornstein–Uhlenbeck process: Mean $m(t)$ and covariance $P(t)$

$$dx = -\lambda x dt + d\beta \tag{6}$$

where $\lambda > 0$ and Brownian motion $\beta(t)$ has diffusion constant q.

$$\frac{dm}{dt} = E[-\lambda x] = -\lambda m \tag{7}$$

$$\frac{dP}{dt} = 2E[-\lambda(x-m)^2] + E[q] = -2\lambda P + q \tag{8}$$

This result leads to the solutions

$$m = m_0 \exp(-\lambda t) \tag{9}$$

$$P = \frac{q}{2\lambda}(1 - \exp(-2\lambda t)) \tag{10}$$

with $P_0 = 0$.

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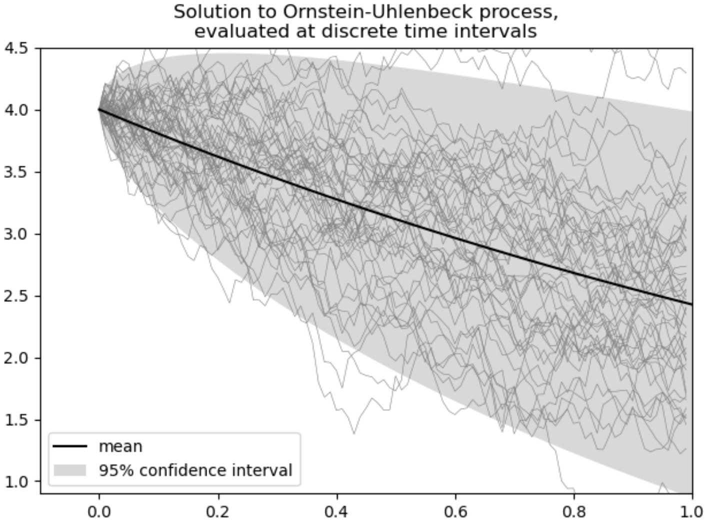

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Because the solution is a linear transformation of Brownian motion, which is a Gaussian process, the solution is Gaussian

$$p(\boldsymbol{x}, t) := p(\boldsymbol{x}(t)) = \mathcal{N}(\boldsymbol{x}(t) | \boldsymbol{m}(t), \boldsymbol{P}(t)), \tag{13}$$

which can be verified by checking that this density indeed solves the corresponding FPK equation.

Ornstein–Uhlenbeck process: Mean $m(t)$ and covariance $P(t)$

$$dx = -\lambda x dt + d\beta \tag{14}$$

What is the form of $\Psi(s,t)$? One of the defining properties of $\Psi(\tau,t)$ is

$$\frac{\partial \Psi(\tau,t)}{\partial \tau} = -F(\tau)\Psi(\tau,t) \tag{15}$$

Let $\Psi(s,t) = \exp(-\int_t^s \lambda(\tau)d\tau)$, which is a solution of Eq. 15 and all the other properties of $\Psi(\tau,t)$. This result leads to the solutions

$$m = m_0 \exp(-\lambda t) \tag{16}$$

$$P = \frac{q}{2\lambda}\left(1 - \exp(-2\lambda t)\right) \tag{17}$$

where $P_0 = 0$

Matrix Fraction Decomposition

In reality, $\boldsymbol{P}(t)$ may not be easy to solve, say if we can't find a form of $\boldsymbol{\Psi}(\tau, t)$.

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A convenient numerical method for solving the covariance from the Lyapunov differential equation. If we defined $\boldsymbol{C}(t)$ and $\boldsymbol{D}(t)$ such that $\boldsymbol{P}(t) = \boldsymbol{C}(t)\boldsymbol{D}^{-1}(t)$ then $\boldsymbol{P}(t)$ solves the matrix Lyapunov equation

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F}(t)\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{F}^T(t) + \boldsymbol{L}(t)\boldsymbol{Q}\boldsymbol{L}^T(t) \tag{18}$$

if the matrices $\boldsymbol{C}(t)$ and $\boldsymbol{D}(t)$ solve the differential equation

$$\begin{bmatrix} d\boldsymbol{C}/dt \\ d\boldsymbol{D}/dt \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}(t) & \boldsymbol{L}(t)\boldsymbol{Q}\boldsymbol{L}^T(t) \\ 0 & -\boldsymbol{F}^T(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{D} \end{bmatrix}, \tag{19}$$

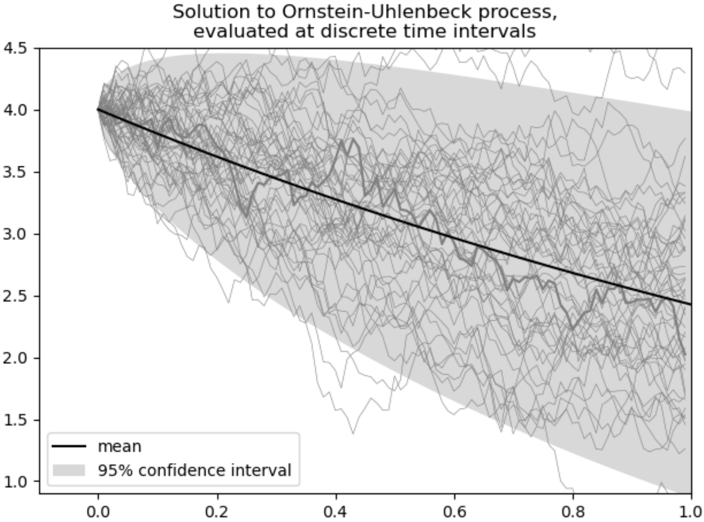
and $\boldsymbol{P}(t_0) = \boldsymbol{C}(t_0)\boldsymbol{D}^{-1}(t_0)$. We can select, for example,

$$\boldsymbol{C}(t_0) = \boldsymbol{P}(t_0) \quad \text{and} \quad \boldsymbol{D}(t_0) = \boldsymbol{I} \tag{20}$$

Obtaining the transition density $p(\boldsymbol{x}(t)|\boldsymbol{x}(s))$

Remember that we solved for the mean $\boldsymbol{m}(t)$ and covariance $\boldsymbol{P}(t)$ using initial conditions $\boldsymbol{m}(t_0) = \boldsymbol{m}_0$ and $\boldsymbol{P}(t_0) = \boldsymbol{P}_0$? The transitional density can be recovered by formally using the initial conditions $\boldsymbol{m}(s) = \boldsymbol{x}(s)$ and $\boldsymbol{P}(s) = 0$

Ornstein–Uhlenbeck process: Transition density $p(x(t)|x(s))$



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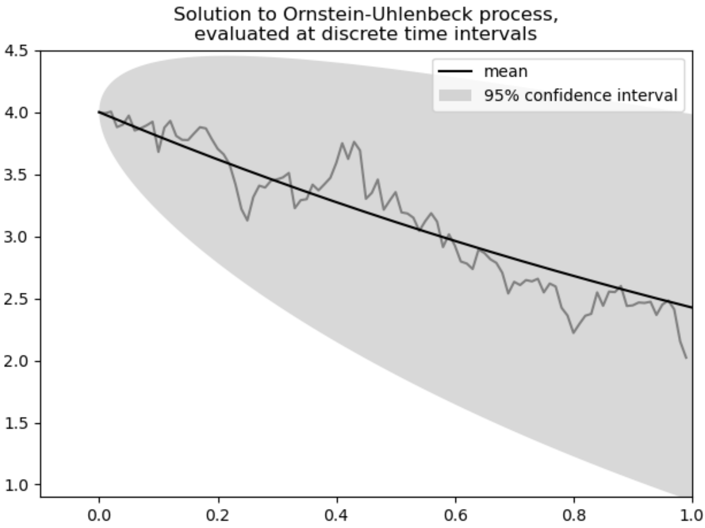

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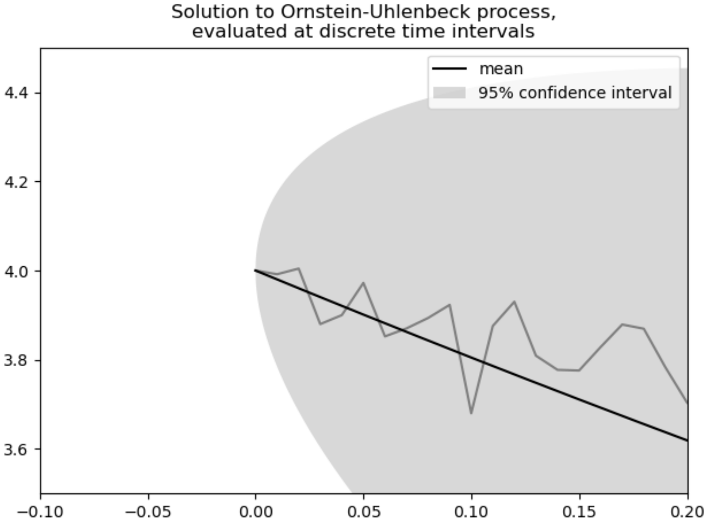

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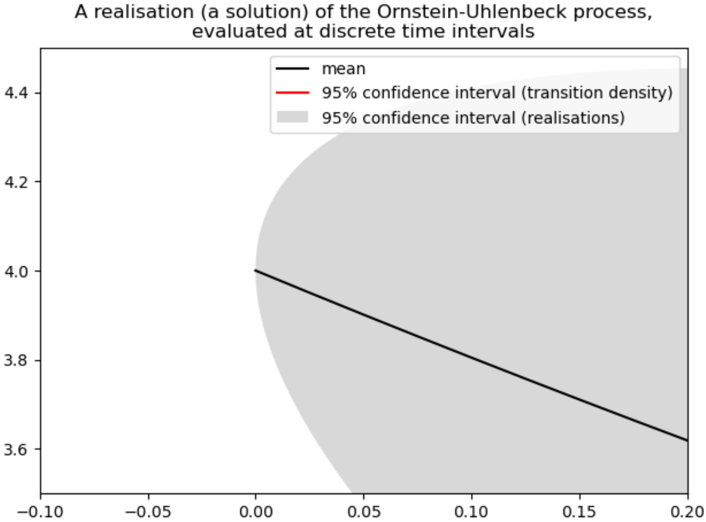

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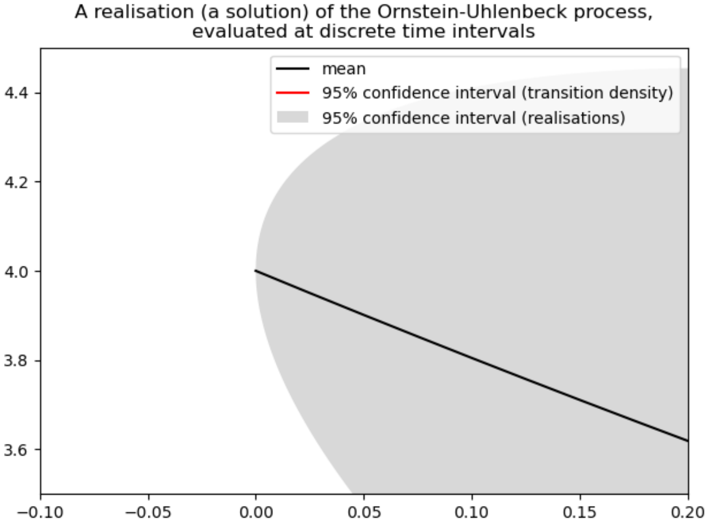

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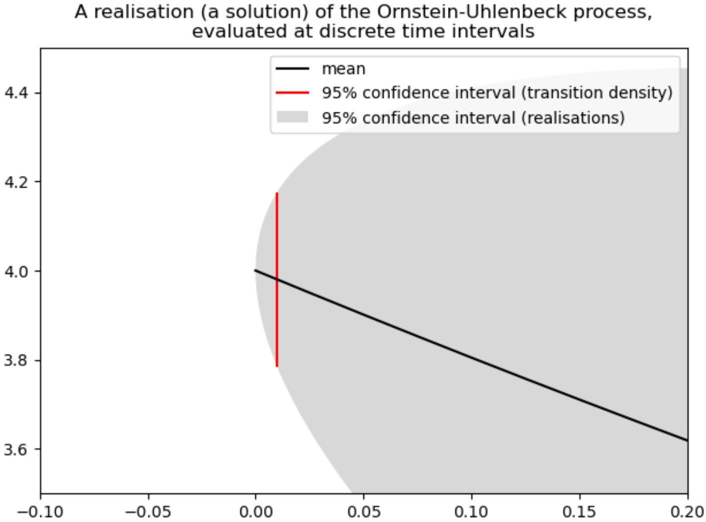

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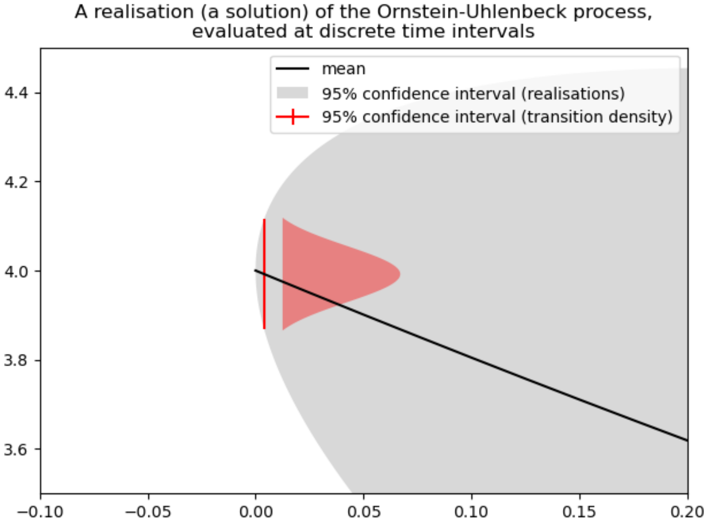

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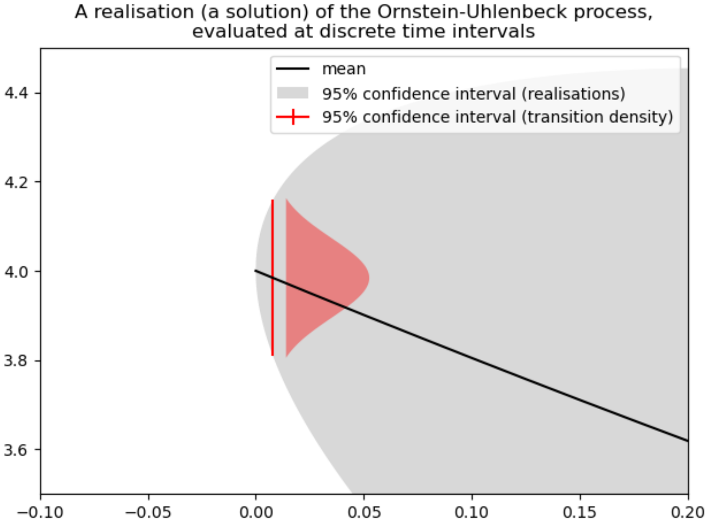

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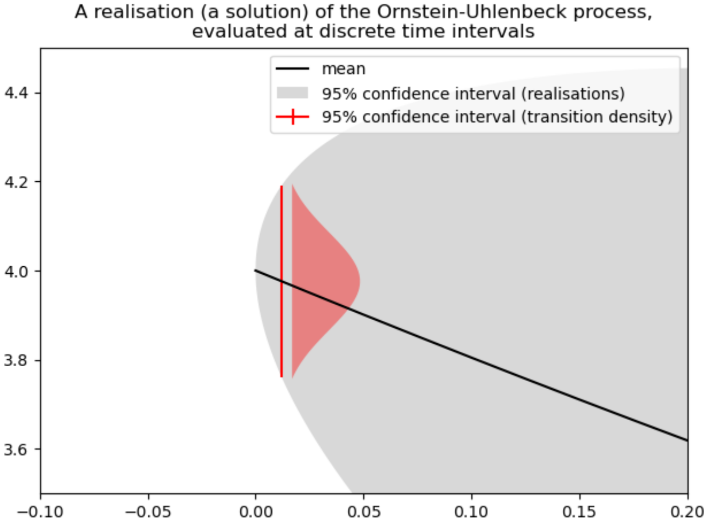

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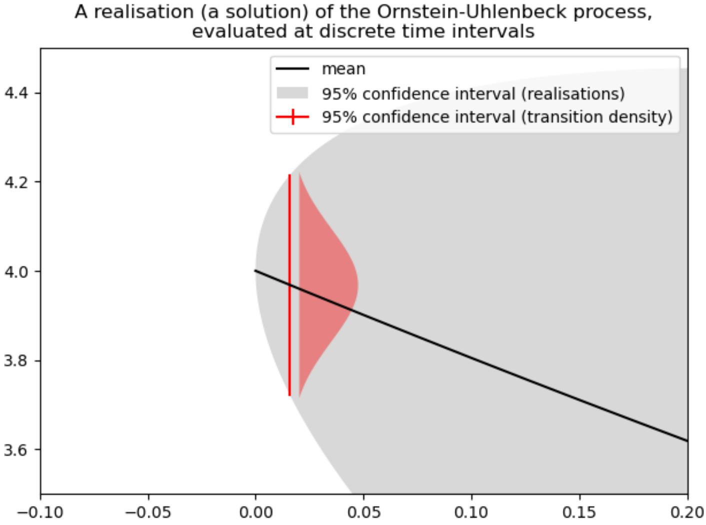

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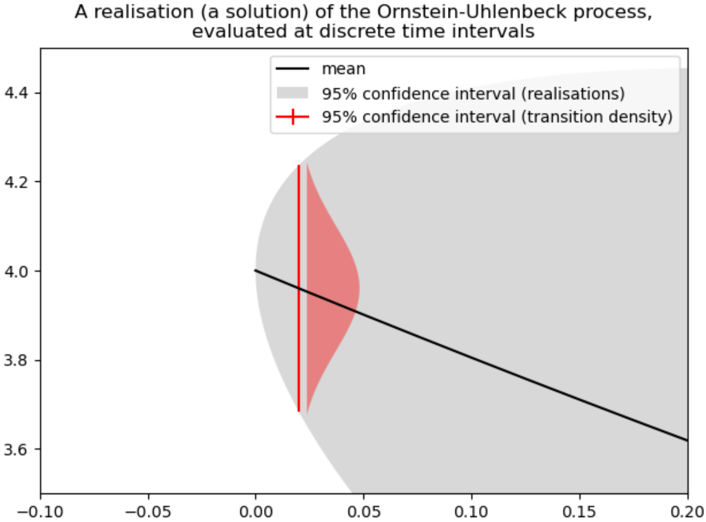

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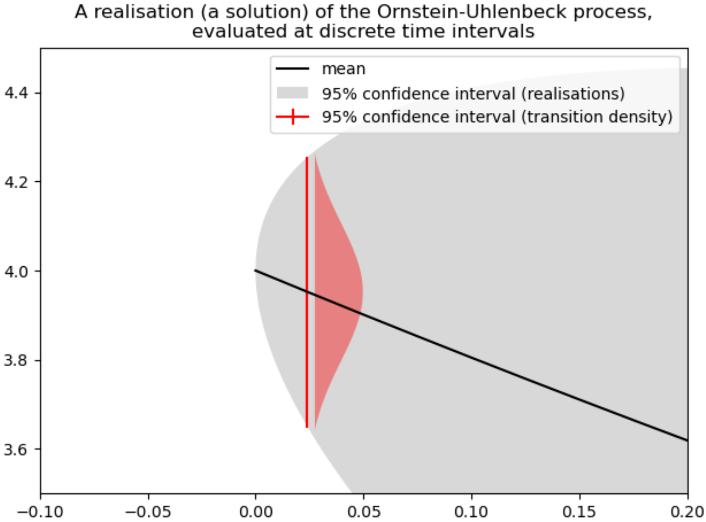

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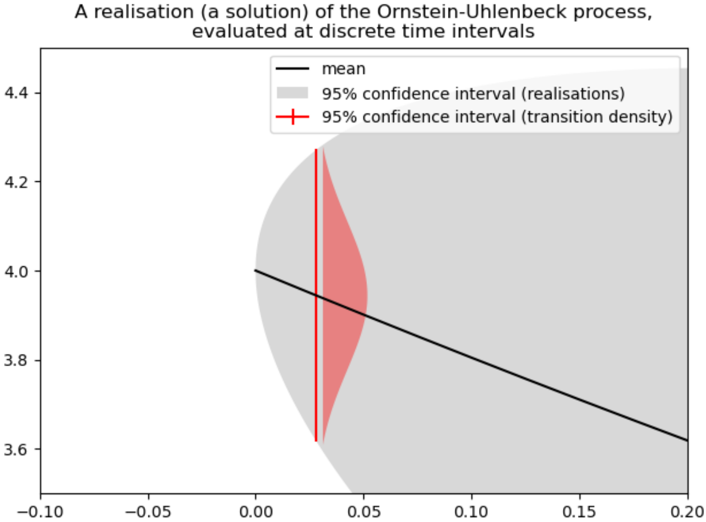

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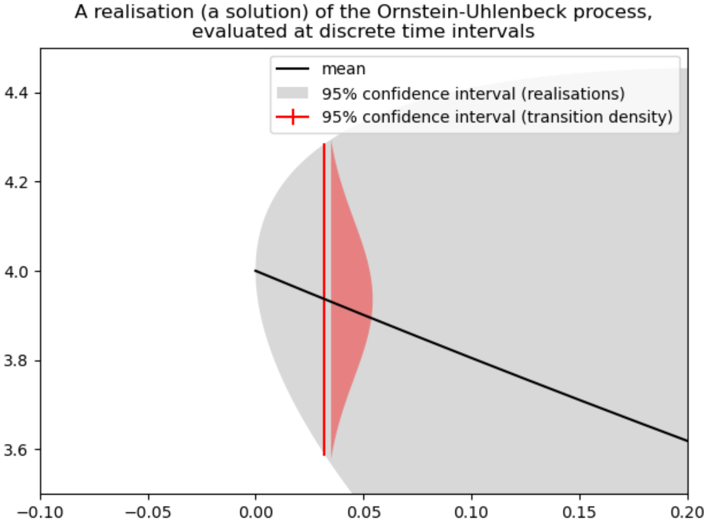

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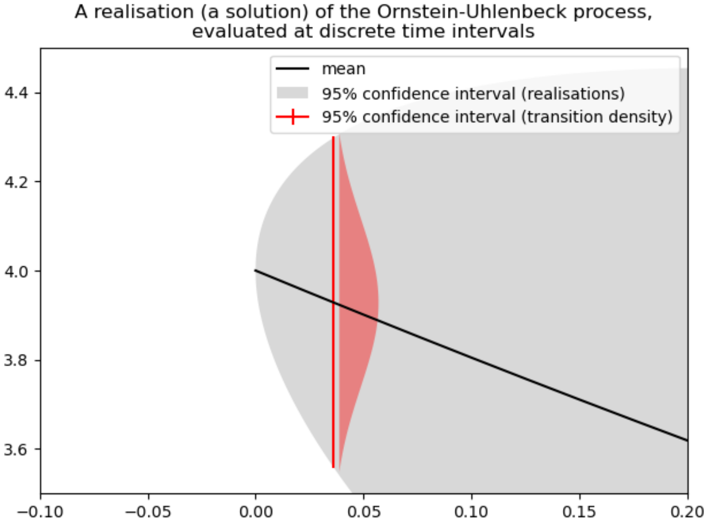

Obtaining the transition density $p(\boldsymbol{x}(t)|\boldsymbol{x}(s))$

$\boldsymbol{m}(s) = \boldsymbol{x}(s)$ and $\boldsymbol{P}(s) = 0$ gives

$$p(\boldsymbol{x}(t)|\boldsymbol{x}(s)) = \mathcal{N}(\boldsymbol{x}(t)|\boldsymbol{m}(t|s), \boldsymbol{P}(t|s)), \tag{21}$$

where

$$\boldsymbol{m}(t|s) = \boldsymbol{\Psi}(t,s)\boldsymbol{x}(s) + \int_s^t \boldsymbol{\Psi}(t,\tau)\boldsymbol{u}(\tau)d\tau, \tag{22}$$

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This implies that the original linear SDE is (weakly, in distribution) equivalent to the following discrete-time system:

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{A}_k\boldsymbol{x}(t_k) + \boldsymbol{u}_k + \boldsymbol{q}_k, \quad \boldsymbol{q}_k \sim \mathcal{N}(0, \boldsymbol{\Sigma}_k), \tag{26}$$

which is sometimes called the *equivalent discretisation of SDEs* in Kalman filtering context.

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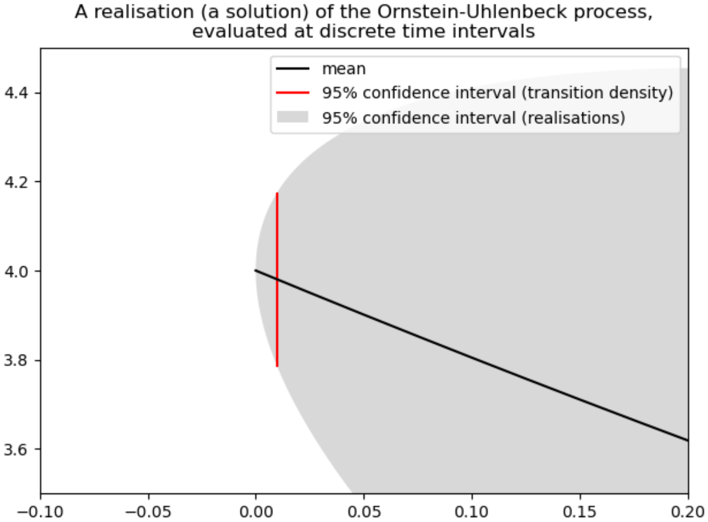

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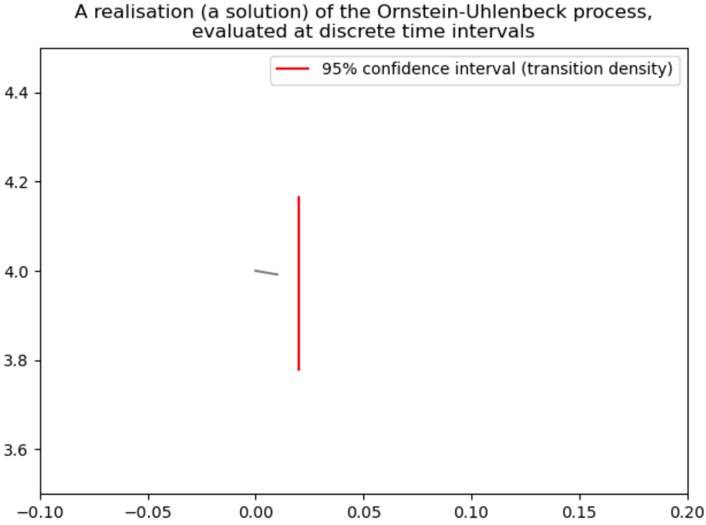

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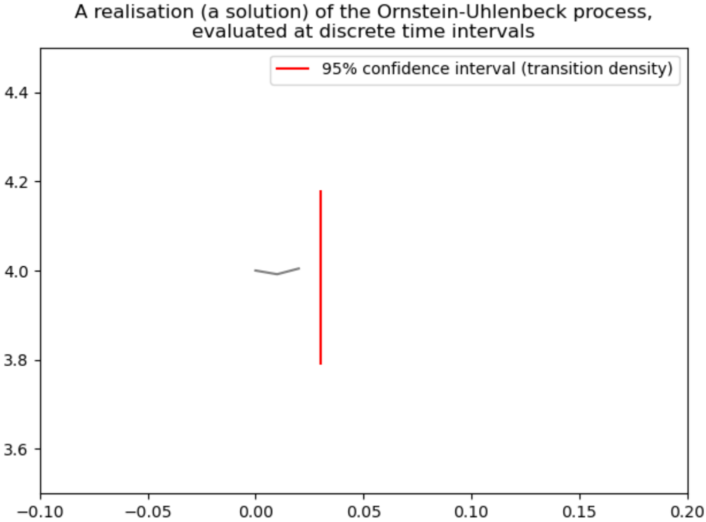

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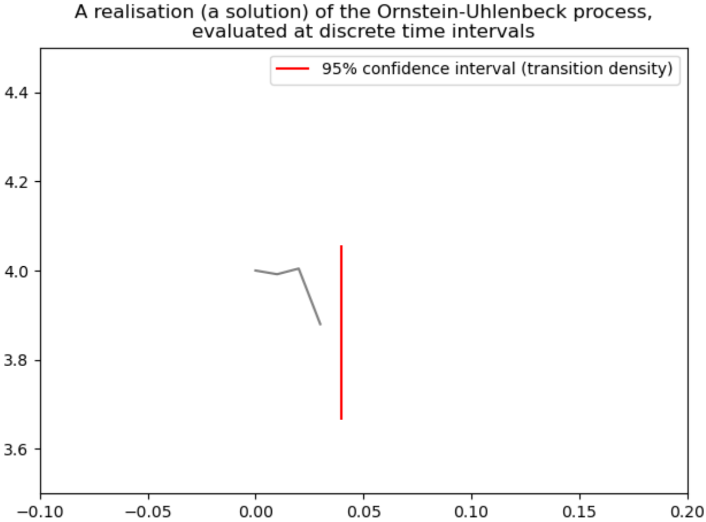

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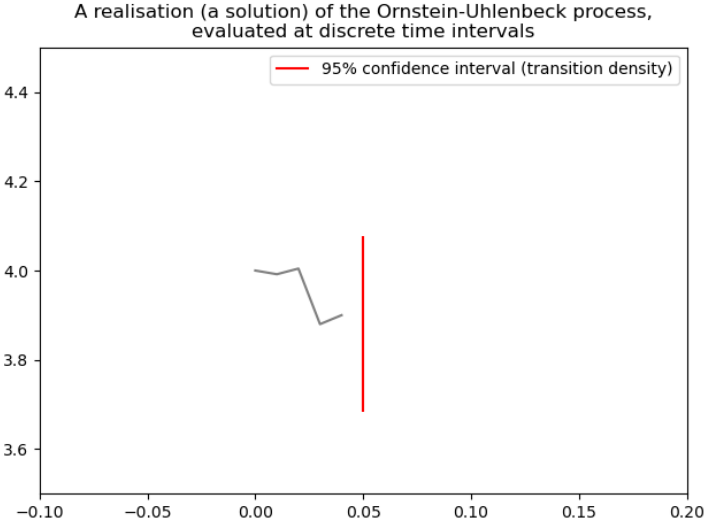

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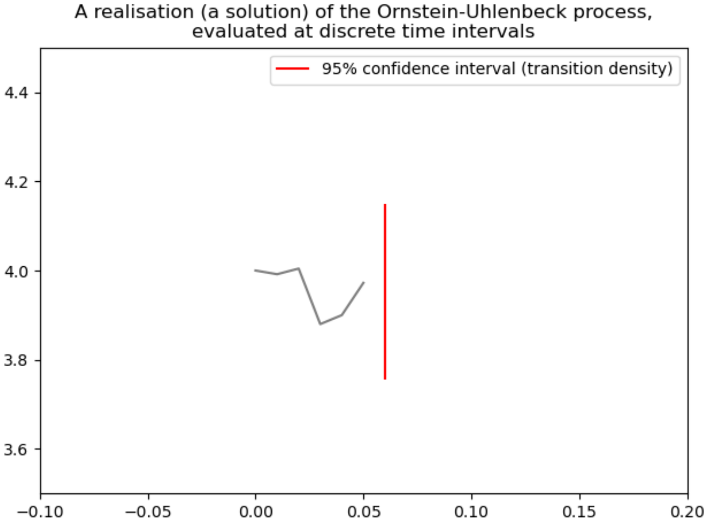

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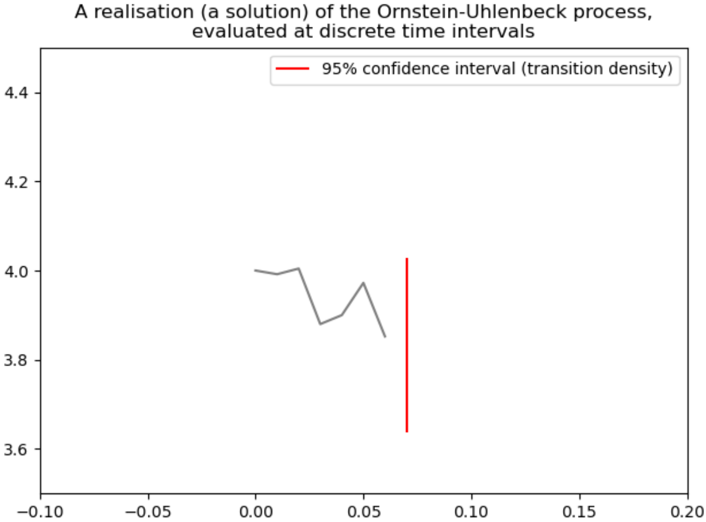

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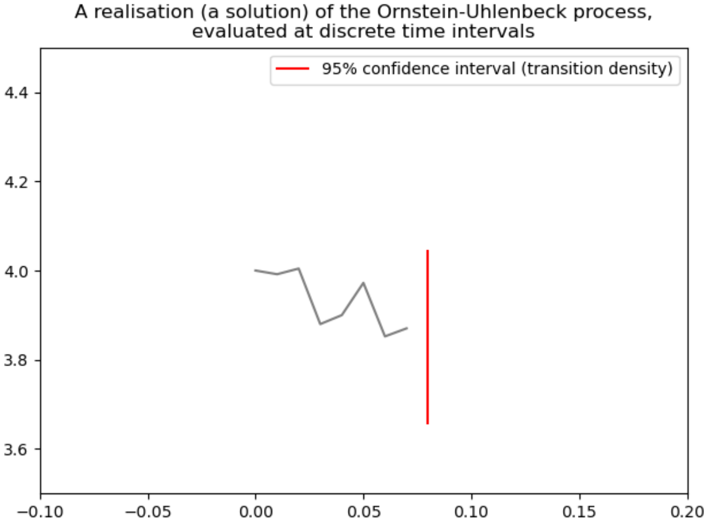

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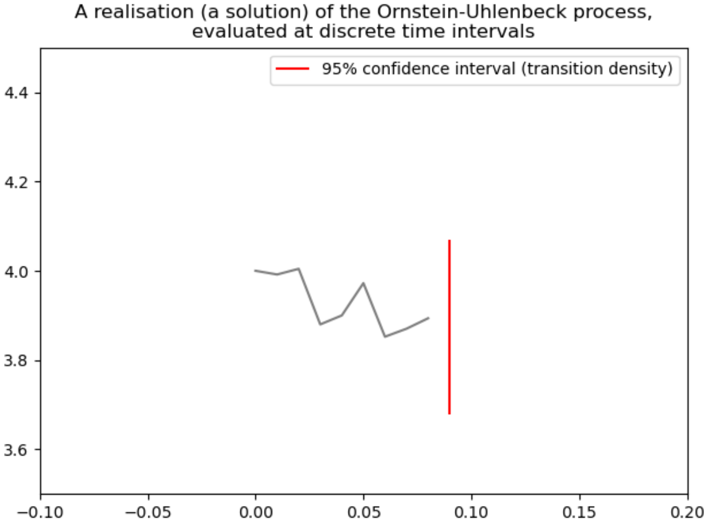

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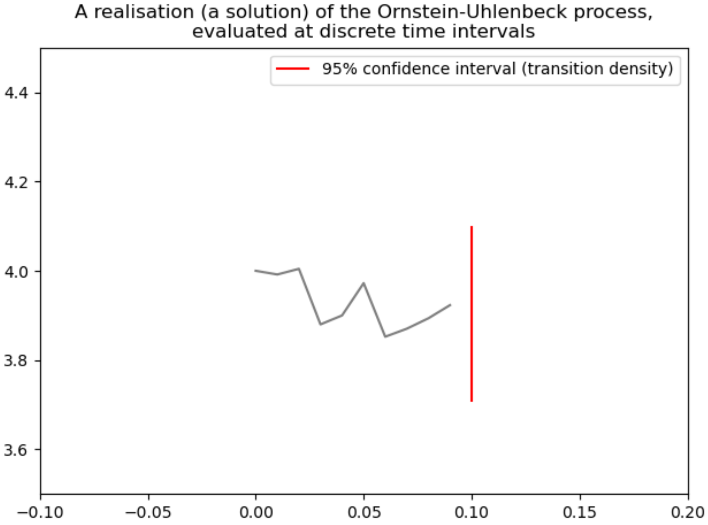
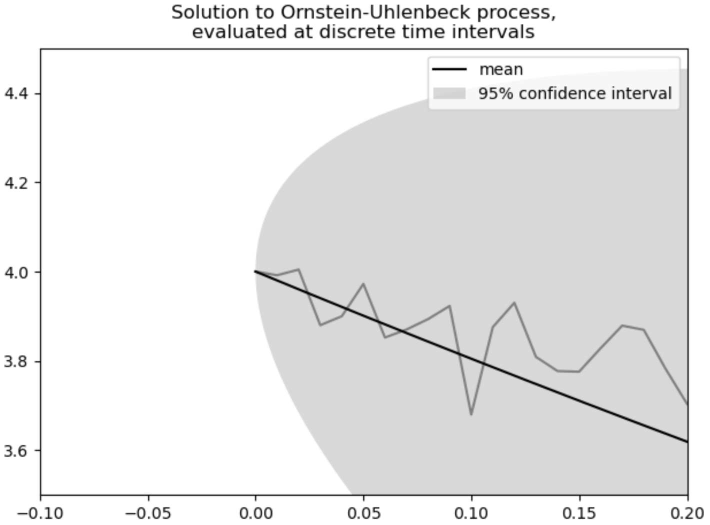


Ornstein–Uhlenbeck process: Equivalent discretisation

At the discrete time-steps $\{t_k\}$, then the distributions of the continuous time and discrete-time equivalent coincide.



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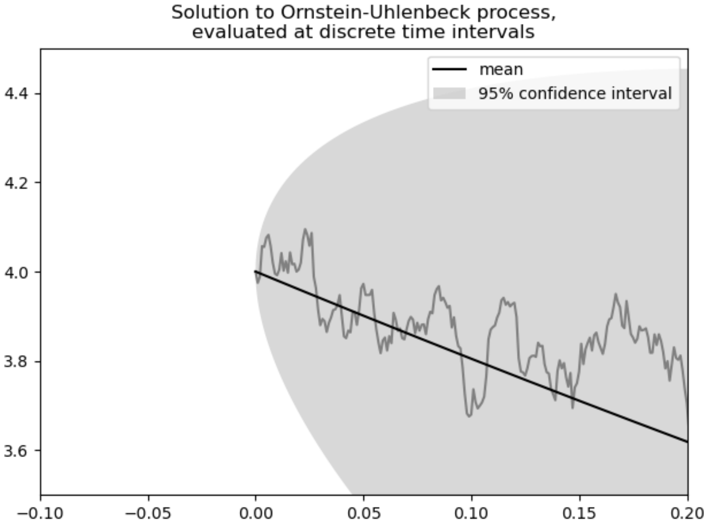

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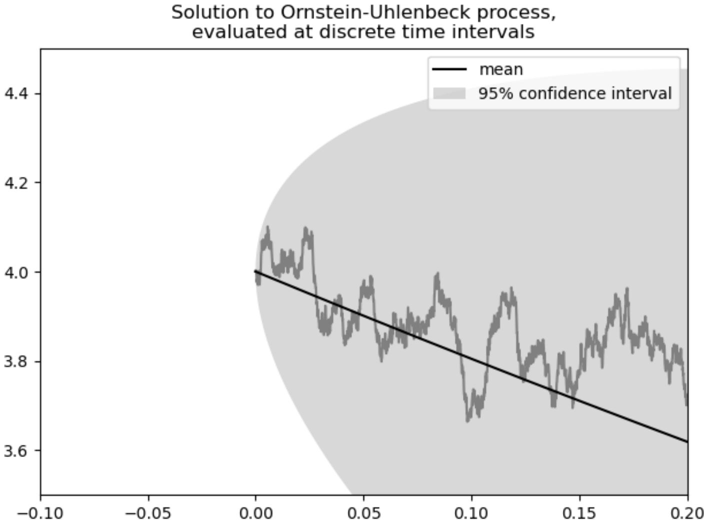

Covariance function of LTI SDEs

The autocovariance function $\boldsymbol{C}(t,s)$ characterises the interaction of states at different times, t and s, it is defined as

$$\boldsymbol{C}(t,s) = E[(\boldsymbol{x}(t) - \boldsymbol{m}(t))(\boldsymbol{x}(s) - \boldsymbol{m}(s))^T] \tag{27}$$

It can be shown that, in the LTI case,

$$\boldsymbol{C}(t,s) \begin{cases} \boldsymbol{P}(t)\exp((s-t)\boldsymbol{F})^T, & \text{if } t < s \\ \exp((t-s)\boldsymbol{F})\boldsymbol{P}(t), & \text{if } t \geq s. \end{cases} \tag{28}$$

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$$\boldsymbol{C}(\tau) \begin{cases} \boldsymbol{P}_\infty \exp(\tau\boldsymbol{F})^T, & \text{if } \tau > 0 \\ \exp(-\tau\boldsymbol{F})\boldsymbol{P}_\infty(t), & \text{if } \tau \leq 0 \end{cases} \tag{29}$$

Fourier Analysis of LTI SDEs

Consider the stochastic differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}\mathbf{x} + \mathbf{L}\mathbf{w} \tag{30}$$

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For a system in its stationary state, then $\boldsymbol{G}(i\omega) = (i\omega\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{L}$ can be thought of as the transfer function of the system.

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This provides a useful means of computing the covariance function of a solution to a stochastic differential equation without first explicitly solving the equation.