Useful Theorems and Formulas for SDEs

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Description

Transformation done by using the substitution $y = \int_{\xi}^{x} \frac{1}{L(u,t)} \mathrm{d}u$ which allows us to change an SDE with multiplicative noise:

$$\mathrm{d}x = f(x,t)\mathrm{d}t + L(x,t)\mathrm{d}\beta \tag{1}$$

Into one with additive noise:

$$\mathrm{d}y = g(y,t)\mathrm{d}t + \mathrm{d}\beta \tag{2}$$

Note that it is possible to extend this to a multivariate setting when $\mathbf{L}(\mathbf{x}, t)$ is diagonal with $L_{ii}(\mathbf{x}, t)$ only depending on x_i.

Approach for Scalar SDE

1. Assuming we have an SDE of the following form:

$$\mathrm{d}x = f(x,t)\mathrm{d}t + L(x,t)\mathrm{d}\beta \tag{3}$$

2. Use Itô's formula to compute $\mathrm{d}y$ given $y = \int_\xi^x \frac{1}{L(u,t)}\mathrm{d}u$:

$$\mathrm{d}y = \underbrace{\left(\frac{\partial}{\partial t}\int_\xi^x \frac{1}{L(u,t)}\mathrm{d}u + \frac{f(x,t)}{L(x,t)} - \frac{1}{2}\frac{\partial L(x,t)}{\partial x}\right)\bigg|_{x=h^{-1}(y,t)}}_{g(y,t)} \mathrm{d}t + \mathrm{d}\beta \tag{4}$$

3. Solve the preceding SDE for $y(t)$ and compute the solution for $x(t)$ by undoing the transformation through a re-substitution.

Example

We want to solve the following SDE:

$$\mathrm{d}x = \left(\alpha x \log x + \frac{1}{2}x\right)\mathrm{d}t + x\ \mathrm{d}\beta \tag{5}$$

Using the substitution $y = \int_{\xi}^{x} \frac{1}{u}\mathrm{d}u = \log(x)$ where $\xi = 1$, we can calculate $\mathrm{d}y$ to get $\mathrm{d}y = \alpha y \mathrm{d}t + \mathrm{d}\beta$. Solving for y(t):

$$y(t) = y(t_0)\exp(\alpha(t - t_0)) + \int_{t_0}^{t} \exp(\alpha(t - \tau))\mathrm{d}\beta(\tau) \tag{6}$$

Re-substituting to get a solution for x(t):

$$x(t) = \exp\left(\log(x(t_0))\exp(\alpha(t - t_0)) + \int_{t_0}^{t} \exp(\alpha(t - \tau))\mathrm{d}\beta(\tau)\right) \tag{7}$$

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Key Definitions

Definition

Brownian motion: a continuous stochastic process $\boldsymbol{\beta}(t) \in \mathbb{R}^S$ that has the following properties:

1. Given $\mathbf{Q}$ is the diffusion matrix of the process and we define $\Delta\boldsymbol{\beta}_k = \boldsymbol{\beta}_{k+1} - \boldsymbol{\beta}_k$ and $\Delta t_k = \Delta t_{k+1} - \Delta t_k$, then $\Delta\boldsymbol{\beta}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}\Delta t_k)$.
2. $\boldsymbol{\beta}(t)$ has independent increments (given no time overlaps).
3. $\boldsymbol{\beta}(0) = \mathbf{0}$.

Definition

Quadratic variation:

$$[X, X]_t = \lim_{t_{k+1} - t_k \to 0} \sum_{t_k \leq t} \left| X_{t_k} - X_{t_{k-1}} \right|^2$$

Lévy's Characterization

Using the first property of Brownian motion β from the previous slide and assuming $q = 1$, we can claim that $[\beta, \beta]_t = t$ which is key for Lévy's characterization.

Theorem

Lévy's characterization of Brownian motion: let the stochastic process $\{X(\tau) | 0 \leq \tau \leq t\}$ have the following properties:

1. $X(t)$ *is a continuous martingale.*
2. $X(0) = 0$.
3. $[X, X]_t = t$.

Then, $X(t)$ can be considered a standard Brownian motion.

Weiner Measure

Using the definition of Brownian motion, we can define the joint distribution over $\beta(t)$ evaluated at a finite number of time points:

$$p\left(\beta\left(t_1\right), \ldots, \beta\left(t_T\right)\right) = \prod_{k=0}^{T-1} \mathcal{N}\left(\beta\left(t_{k+1}\right) \mid \beta\left(t_k\right), t_{k+1} - t_k\right) \quad (8)$$

This is valid for any finite number of time points and therefore defines a valid probability measure for the stochastic process. It is also possible to show that this measure has several properties such as that this measure is unique and has a continuous version.

Weiner Measure via Path Integral

We can use this to define a measure over a set $\mathcal{B}_T$ of discrete paths:

$$
\begin{aligned}
& P\left(\left(\beta\left(t_1\right), \ldots, \beta\left(t_T\right)\right) \in \mathcal{B}_T\right) \\
& = \int_{\mathcal{B}_T} p\left(\beta\left(t_1\right), \ldots, \beta\left(t_T\right)\right) \mathrm{d}\beta\left(t_1\right) \times \cdots \times \mathrm{d}\beta\left(t_T\right) \\
& = \int_{\mathcal{B}_T} \exp\left(-\tfrac{1}{2} \sum_{k=0}^{T-1} \tfrac{\left(\beta\left(t_{k+1}\right)-\beta\left(t_k\right)\right)^2}{\left(t_{k+1}-t_k\right)^2}\left(t_{k+1}-t_k\right)\right) \\
& \quad \times \prod_{k=0}^{T-1} \tfrac{\mathrm{d}\beta\left(t_{k+1}\right)}{\sqrt{2\pi\left(t_{k+1}-t_k\right)}}
\end{aligned} \tag{9}
$$

Given $\mathcal{B}$ is the set of functions on $[0, t]$ and that $\left(\beta\left(t_{k+1}\right)-\beta\left(t_k\right)\right)^2 / \left(t_{k+1}-t_k\right)^2 \rightarrow \dot{\beta}^2(t)$:

$$
P_{\mathrm{W}}(\beta \in \mathcal{B}) = \int_{\mathcal{B}} \exp\left(-\frac{1}{2} \int_0^t \dot{\beta}^2(\tau) \mathrm{d}\tau\right) \prod_{\tau=0}^{t} \frac{\mathrm{d}\beta(\tau)}{\sqrt{2\pi \mathrm{d}\tau}} \tag{10}
$$

Limit of a Random Walk

We may construct Brownian motion by considering the limit of a random walk time increment tends to zero. A random walk is a stochastic process that changes value at each integer step. For example:

$$\beta_{k+1} = \beta_k + q_k, \quad q_k \sim \mathcal{N}(0, t_{k+1} - t_k) \tag{11}$$

Let $s_k = \sum_{i=1}^{k} \xi_i$ where $\xi \in \{-1, 1\}$ and $S_n(t) = \frac{s_{[nt]}}{\sqrt{n}}$. We can then define the following:

$$S_n\left(t_{k+1}\right) - S_n\left(t_k\right) = \frac{S_{[nt_{k+1}]}}{\sqrt{n}} - \frac{S_{[nt_k]}}{\sqrt{n}} = \frac{\sum_{i=nt_k+1}^{nt_{k+1}} \xi_i}{\sqrt{n}} \tag{12}$$

Limit of a Random Walk (Cont.)

Theorem

Central limit theorem: given the random variable X_i where $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[{X_i}^2] = \sigma^2$, we may claim that $\frac{\sum_{i=0}^{n} X_i}{\sqrt{n}} \to \mathcal{N}(0, \sigma^2)$ as $n \to \infty$.

$$\mathbb{E}[S_n(t_{k+1}) - S_n(t_k)] = \frac{\sum_{i=nt_k+1}^{nt_{k+1}} \mathbb{E}[\xi_i]}{\sqrt{n}} = 0 \tag{13}$$

$$V[S_n(t_{k+1}) - S_n(t_k)] = \frac{\sum_{i=nt_k+1}^{nt_{k+1}} V[\xi_i]}{n} \tag{14}$$

$$= \frac{nt_{k+1} - nt_k - 1 + 1}{n} \tag{15}$$

$$= t_{k+1} - t_k \tag{16}$$

Therefore, as $n \to \infty$, we may claim that:

$$S_n(t_{k+1}) - S_n(t_k) \to \mathcal{N}(0, t_{k+1} - t_k) \tag{17}$$

$$S_n(t) \to \beta(t) \tag{18}$$

Karhunen-Loeve Expansion

Brownian motion can also be defined as a zero-mean Gaussian process with the covariance function $C(t, t') = \min(t, t')$. We may therefore use Mercer's theorem which states that for eigenvalues λ_n and eigenfunctions ϕ_n, we may perform the following decomposition:

$$C\left(t, t'\right) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t) \phi_n\left(t'\right) \tag{19}$$

We can therefore describe Brownian motion through its karhunen-Loeve expansion, where $z_n \sim \mathcal{N}(0, \lambda_n)$:

$$\beta(t) = \sum_{n=1}^{\infty} z_n \phi_n(t) \tag{20}$$

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Purposes

The importance of the Girsanov theorem cannot be overstate. Notable use cases include:

1. Transforming a probability measure of SDEs.
2. Removing and transforming drift function of SDEs.
3. Finding weak solutions to SDEs.
4. Used as a starting point to derive the Kallianpur–Striebel formula (Bayes' rule in continuous time).
5. Form MC methods for approximating filtering solutions.
6. Construct sampling methods for conditioned SDEs.

Definitions

Let $\mathbf{x}(t)$ be stochastic process which solves the SDE $\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\mathrm{d}t + \mathrm{d}\boldsymbol{\beta}$ with the path on time interval [0, t] denoted as $\mathcal{X}_t = \{\mathbf{x}(\tau) | 0 \leq \tau \leq t\}$.

$$p\left(\mathcal{X}_t\right) = \lim_{n \to \infty} p\left(\mathbf{x}\left(t_1\right), \mathbf{x}\left(t_2\right), \ldots, \mathbf{x}\left(t_n\right)\right) \tag{21}$$

$$\frac{p\left(\mathcal{X}_t\right)}{p\left(\mathcal{Y}_t\right)} = \lim_{n \to \infty} \frac{p\left(\mathbf{x}\left(t_1\right), \mathbf{x}\left(t_2\right), \ldots, \mathbf{x}\left(t_n\right)\right)}{p\left(\mathbf{y}\left(t_1\right), \mathbf{y}\left(t_2\right), \ldots, \mathbf{y}\left(t_n\right)\right)} \tag{22}$$

$$\mathrm{E}\left[h\left(\mathcal{X}_t\right)\right] = \int h\left(\mathcal{X}_t\right) p\left(\mathcal{X}_t\right) \mathrm{d}\mathcal{X}_t \tag{23}$$

Likelihood Ratio

Theorem

Likelihood ratio of Itô process: Let $\mathrm{x}(t)$ *and* $\mathrm{y}(t)$ *respectively solve the SDEs* $\mathrm{d}\mathrm{x} = \mathrm{f}(\mathrm{x}, t)\mathrm{d}t + \mathrm{d}\boldsymbol{\beta}$ *and* $\mathrm{d}\mathrm{y} = \mathrm{g}(\mathrm{y}, t)\mathrm{d}t + \mathrm{d}\boldsymbol{\beta}$*. Given that* $\frac{p(\mathcal{X}_t)}{p(\mathcal{Y}_t)} = Z(t)$*, we may claim the following:*

$$
\begin{aligned}
Z(t) = \exp\Bigg(-\frac{1}{2}\int_0^t [\mathrm{f}(\mathrm{y}, \tau) - \mathrm{g}(\mathrm{y}, \tau)]^\top \mathrm{Q}^{-1} [\mathrm{f}(\mathrm{y}, \tau) - \mathrm{g}(\mathrm{y}, \tau)]\, \mathrm{d}\tau \\
+ \int_0^t [\mathrm{f}(\mathrm{y}, \tau) - \mathrm{g}(\mathrm{y}, \tau)]^\top \mathrm{Q}^{-1}\, \mathrm{d}\boldsymbol{\beta}(\tau) \Bigg)
\end{aligned}
\tag{24}
$$

Likelihood Ratio (Cont.)

Theorem (Cont.)

For an arbitrary function h, we may claim that:

$$\mathbb{E}[h(\mathcal{X}_t)] = \mathbb{E}[Z(t)h(\mathcal{Y}_t)] \tag{25}$$

Furthermore, given the probability density $\tilde{p}\left(\mathcal{X}_t\right) = Z(t)p\left(\mathcal{X}_t\right)$, the following process is a Brownian motion with diffusion matrix Q under the transformed probability density:

$$\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta} - \int_0^t [\mathbf{f}(\mathbf{y}, \tau) - \mathbf{g}(\mathbf{y}, \tau)] \mathrm{d}\tau \tag{26}$$

Girsanov I

Theorem

Girsanov I: Let $\boldsymbol{\theta}(t)$ be a process that is driven by standard Brownian motion $\beta(t)$ such that $\mathbb{E}[\int_0^t \boldsymbol{\theta}^T(\tau)\boldsymbol{\theta}(\tau)\mathrm{d}\tau] < \infty$. Given the new measure defined by $\tilde{p}(\Theta_t) = Z(t)p(\Theta_t)$ where

$$Z(t) = \exp\left(\int_0^t \boldsymbol{\theta}^\top(\tau)\mathrm{d}\boldsymbol{\beta} - \frac{1}{2}\int_0^t \boldsymbol{\theta}^\top(\tau)\boldsymbol{\theta}(\tau)\mathrm{d}\tau\right) \tag{27}$$

Then the following process is a standard Brownian motion:

$$\tilde{\boldsymbol{\beta}}(t) = \boldsymbol{\beta}(t) - \int_0^t \boldsymbol{\theta}(\tau)\mathrm{d}\tau \tag{28}$$

Example of Weak solution for SDE

Given a stochastic process $\boldsymbol{x}(t)$ which solves $\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\mathrm{d}t + \mathrm{d}\boldsymbol{\beta}$ where $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and we define the following:

$$Z(t) = \exp\left(-\frac{1}{2}\int_0^t \mathbf{f}^\top\left(\mathbf{x}_0 + \boldsymbol{\beta}(\tau), \tau\right)\mathbf{Q}^{-1}\mathbf{f}\left(\mathbf{x}_0 + \boldsymbol{\beta}(\tau), \tau\right)\mathrm{d}\tau \right.$$
$$\left. + \int_0^t \mathbf{f}^\top\left(\mathbf{x}_0 + \boldsymbol{\beta}(\tau), \tau\right)\mathbf{Q}^{-1}\,\mathrm{d}\boldsymbol{\beta}(\tau)\right) \tag{29}$$

We may then claim the following:

$$\mathrm{E}[\mathbf{h}(\mathbf{x}(t))] = \mathrm{E}\left[Z(t)\mathbf{h}\left(\mathbf{x}_0 + \boldsymbol{\beta}(t)\right)\right] \tag{30}$$
$$\tilde{\mathbf{x}}(t) = \mathbf{x}_0 + \boldsymbol{\beta}(t) \tag{31}$$
$$\tilde{\boldsymbol{\beta}}(t) = \boldsymbol{\beta}(t) - \int_0^t \mathbf{f}\left(\mathbf{x}_0 + \boldsymbol{\beta}(\tau)\right)\mathrm{d}\tau \tag{32}$$

Intuition

We first define the following two discrete processes which respectively represent Brownian motion without and with drift:

$$\beta(t_k) = \beta(t_{k-1}) + \Delta\beta(t_k), \quad \Delta\beta(t_k) \sim \mathcal{N}(0, \Delta t q) \tag{33}$$

$$x(t_k) = x(t_{k-1}) + f_k \Delta t + \beta(t_k), \quad \Delta\beta(t_k) \sim \mathcal{N}(0, \Delta t q) \tag{34}$$

We can respectively write the joint distribution for both of these terms using the Weiner measure:

$$p(\beta(t_1), \ldots, \beta(t_n)) = \frac{1}{(\sqrt{2\pi}\Delta t q)^n} \exp\left(-\frac{1}{2\Delta t q} \sum_{k=1}^{n} \Delta\beta_k^2 \right) \tag{35}$$

$$\begin{aligned} p(x(t_1), \ldots, x(t_n)) =& \frac{1}{(\sqrt{2\pi\Delta t q})^n} \exp\left(-\frac{1}{2\Delta t q} \sum_{k=1}^{n} \Delta\beta_k^2 + \right. \\ & \left. \frac{1}{q} \sum_{k=1}^{n} f_k \Delta\beta_k - \frac{1}{2q} \sum_{k=1}^{n} f_k^2 \Delta t \right) \end{aligned} \tag{36}$$

Intuition (Cont.)

The ratio of these discrete probability densities has the following form:

$$\frac{p\left(x\left(t_1\right),\ldots,x\left(t_k\right)\right)}{p\left(\beta\left(t_1\right),\ldots,\beta\left(t_n\right)\right)} = \exp\left(\frac{1}{q}\sum_{k=1}^{n} f_k\ \Delta\beta_k - \frac{1}{2q}\sum_{k=1}^{n} f_k^2 \Delta t\right) \triangleq Z_n\left(\beta\left(t_1\right),\ldots,\beta\left(t_n\right)\right), \tag{37}$$

When we take the limit as $n \to \infty$ we get the following:

$$Z(t) = \exp\left(\frac{1}{q}\int_0^t f(t)\mathrm{d}\beta - \frac{1}{2q}\int_0^t f^2(t)\mathrm{d}t\right) \tag{38}$$

Note that this still works if f depends on the process β.

Intuition (Cont.)

Since Z_n is just a ratio of densities, we may claim:

$$\mathbb{E}[h(x(t_1), \ldots, x(t_n)] = \mathbb{E}[Z_n(\beta(t_1), \ldots, \beta(t_n))h(\beta(t_1), \ldots, \beta(t_n))] \tag{39}$$

Hence, Z_n allows us to construct a transformed probability measure:

$$\tilde{p}\left(\beta\left(t_1\right), \ldots, \beta\left(t_n\right)\right) = Z_n\left(\beta\left(t_1\right), \ldots, \beta\left(t_k\right)\right) p\left(\beta\left(t_1\right), \ldots, \beta\left(t_n\right)\right) \tag{40}$$

Similar to the continuous case, we can define the following process which is a Brownian motion defined under $\tilde{p}$:

$$\tilde{\beta}(t_k) = \beta(t_k) - \sum_k f_k \Delta t \tag{41}$$

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Purposes

Several use cases of Doob's h-transform include:

1. Deriving an SDE conditioned on another SDE at its end point.
2. Removing drift from SDEs.
3. Analyzing hitting times and excursions of SDEs.

Definition

Hitting time: the first time at which a stochastic process assumes some value in a subset of the sample space.
Excursion probability: the probability that the stochastic process surpasses some value during a fixed time period.

Important Terms

Given a transition density defined as $p(\mathbf{y}, t'|\mathbf{x}, t) \triangleq p(\mathbf{y}(t')|\mathbf{x}(t))$, we may define a function that satisfies the following (space-time regularity property):

$$h(t, \mathbf{x}) = \int p(\mathbf{y}, t'|\mathbf{x}, t) h(t + s, \mathbf{y}) \mathrm{d}\mathbf{y} \tag{42}$$

Using such a function, we can define a new Markov process with the transition kernel:

$$p^h(\mathbf{y}, t + s|\mathbf{x}, t) = p(\mathbf{y}, t + s|\mathbf{x}, t) \frac{h(t + s, \mathbf{y})}{h(t, \mathbf{x})} \tag{43}$$

Intuitive Overview

The following steps outlines the mechanics behind Doob's h-transform:

1. We are interested in constructing an SDE that follows $p^h = p(\mathbf{x}(t+s)|\mathbf{x}(t), \mathbf{x}(T))$.
2. Choosing $h(t, \mathbf{x}) = p(\mathbf{x}(T)|\mathbf{x}(t))$ allows the p^h to be the desired distribution and h also satisfies the space-time regularity property.
3. Use the operator $\mathcal{A}$ on $\Phi(\mathbf{x})$ where $\mathbf{x}$ is characterized by p^h. The purposed of doing this is to get the SDE which will follow the distribution that is specified by P^h.

Operator Result

We are able to derive the following:

$$
\begin{aligned}
\mathcal{A}^h\{\Phi(\mathbf{x})\} =& \mathcal{A}\{\Phi(\mathbf{x})h(\mathbf{x})\} \\
=& \sum_i \left[f_i(\mathbf{x},t) + \mathbf{L}(\mathbf{x},t)\mathbf{Q}\mathbf{L}^\top(\mathbf{x},t)\frac{\nabla h(t,\mathbf{x})}{h(t,\mathbf{x})} \right] \frac{\partial \phi(\mathbf{x})}{\partial x_i} \\
&+ \frac{1}{2}\sum_{i,j} \frac{\partial^2 \phi(\mathbf{x})}{\partial x_i \partial x_j} \left[\mathbf{L}(\mathbf{x},t)\mathbf{Q}\mathbf{L}^\top(\mathbf{x},t) \right]_{ij}
\end{aligned} \tag{44}
$$

Where could construct the associated SDE by matching with:

$$
\begin{aligned}
\mathcal{A}_t(\bullet) = \frac{\partial(\bullet)}{\partial t} + \sum_i \frac{\partial(\bullet)}{\partial x_i} f_i(\mathbf{x},t) \\
+ \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2(\bullet)}{\partial x_i \partial x_j}\right)\left[\mathbf{L}(\mathbf{x},t)\mathbf{Q}\mathbf{L}^\top(\mathbf{x},t)\right]_{ij}
\end{aligned} \tag{45}
$$

End point Conditioning

Theorem

Given a process $\mathrm{x}(t)$ *that solves* $\mathrm{d}x = f(x,t)\mathrm{d}t + L(x,t)\mathrm{d}\beta$ *and assuming that we want to condition its solution to hit* $\mathrm{x}(T)$ *at time* $t = T$*, then the h-transform provides us with the following SDE for the conditioned process:*

$$\begin{aligned}\mathrm{d}\mathsf{x} = \Big[\mathsf{f}(\mathsf{x}, t) + \mathsf{L}(\mathsf{x}, t)\mathsf{Q}\mathsf{L}^\top(\mathsf{x}, t)\nabla \log p(\mathsf{x}(T) \mid \mathsf{x}(t))\Big]\,\mathrm{d}t \\ + \mathsf{L}(\mathsf{x}, t)\mathrm{d}\boldsymbol{\beta},\end{aligned} \tag{46}$$

Note that in the statement of this theorem we are using:

$$h(t, \mathbf{x}) = p(\mathbf{x}(T)|\mathbf{x}(t)) \tag{47}$$

Example

Assume we have a process $x(t)$ which solves $\mathrm{d}x = -\lambda x \mathrm{d}t + \mathrm{d}\beta$ where $x(0) = 0$ (Ornstein-Uhlenbeck process). Using the preceding theorem, we want to condition on $x(T) = x_T$:

$$h(t, x) = \mathcal{N}\left(x_T \mid a(t)x, \sigma^2(t)\right) \tag{48}$$

$$a(t) = \exp(-\lambda(T - t)) \tag{49}$$

$$\sigma^2(t) = \frac{q}{2\lambda}[1 - \exp(-2\lambda(T - t))] \tag{50}$$

Which then results in:

$$\mathrm{d}x = \left[-\lambda x + \frac{qa(t)}{\sigma^2(t)}\left(x_T - a(t)x\right)\right]\mathrm{d}t + \mathrm{d}\beta \tag{51}$$

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Path Integrals

The expectation of a functional $F[\beta]$ with respect to its measure could be defined as follows:

$$\mathrm{E}[F] = \int F[\beta] \exp\left(-\frac{1}{2}\int_0^t \dot{\beta}^2(\tau)\mathrm{d}\tau\right) \prod_{\tau=0}^{t} \frac{\mathrm{d}\beta(\tau)}{\sqrt{2\pi \mathrm{d}\tau}} \tag{52}$$

We may rely on SDE methods to solve this. For example, say we want to compute the path integral of $F[\beta] = \exp\left(\int_0^t \beta(s)\mathrm{d}s\right)$ and we know that $x_1(t) = \int_0^t \beta(s)\mathrm{d}s$ is the solution to $\mathrm{d}x_1 = x_2\mathrm{d}t$, $\mathrm{d}x_2 = \mathrm{d}\beta$. Armed with the knowledge that $x_1 \sim \mathcal{N}(0, t^3/3)$, we may compute the following:

$$\mathrm{E}[F] = \int \exp(x_1)\,\mathcal{N}\left(x_1 \mid 0, t^3/3\right)\mathrm{d}x_1 = \exp\left(\frac{t^3}{6}\right) \tag{53}$$

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Description

The Feynman-Kac formula provides a way to link the solution of PDEs with certain expected values of SDE solutions. To demonstrate how this is done, we start off with the following PDE:

$$\begin{aligned}\frac{\partial u}{\partial t} + f(x)\frac{\partial u}{\partial x} + \frac{1}{2}L^2(x)\frac{\partial^2 u}{\partial x^2} &= 0 \\ u(x, T) &= \Psi(x)\end{aligned} \tag{54}$$

We then define a stochastic process $x(t)$ on interval $[t', T]$ as being a solution to $\mathrm{d}x = f(x)\mathrm{d}t + \mathrm{d}\beta$ with $x(t') = x'$. How could we use this process to help us compute the $u(x, T)$?

Description (Cont.)

First, we compute the differential $\mathrm{d}u$ using Itô's formula:

$$\mathrm{d}u = \frac{\partial u}{\partial x} L(x) \mathrm{d}\beta \tag{55}$$

Integrate from t' to T:

$$u(x(T), T) - u(x(t'), t') = \int_{t'}^{T} \frac{\partial u}{\partial x} L(x) \mathrm{d}\beta \tag{56}$$

Take expectations on both sides:

$$u(x(t'), t') = \mathbb{E}[u(x(T), T)] = \mathbb{E}[\Psi(x(T))] \tag{57}$$

Idea: we could solve for $u(x', t')$ by starting a process $x(t)$ from time t' until T and then computing the expectation $\mathbb{E}[\Psi(x(T))]$.

Main Result

This idea could be generalized to solve PDEs of the following form:

$$\frac{\partial u}{\partial t} + f(x)\frac{\partial u}{\partial x} + \frac{1}{2}L^2(x)\frac{\partial^2 u}{\partial x^2} - ru = 0 \tag{58}$$
$$u(x, T) = \Psi(x),$$

In this case, we must use the following Feynman-Kac equation:

$$u\left(x', t'\right) = \exp\left(-r\left(T - t'\right)\right)\mathrm{E}[\Psi(x(T))] \tag{59}$$

The Feynman-Kac formula can be generalized to the multidimensional case and can be used to construct algorithms for the following:

1. Solving Backward PDEs with SDE Simulation.
2. Solving Forward PDEs with SDE Simulation.
3. Solving boundary value problems with SDE simulation.

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Overview

1. Lamperti transform: substitution used to turn an SDE with multiplicative noise into one with additive noise.
2. Brownian motion construction: there exists different ways of constructing Brownian motion (e.g. Lévy, random walk, Karhunen-Loeve).
3. Girsanov theorem: used to perform a change of measure for stochastic processes.
4. Doob's h-transform: used to condition stochastic processes to hit a certain value at a particular time.
5. Path integrals: allows for the use of SDE theory to compute the expectation of functionals.
6. Feynman-Kac formula: provides a link between PDE solutions and expectations of certain SDE solutions.