Chapter 5 - Probability distributions and statistics of SDEs

CSML Reading group

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Outline

1. Important definitions
2. SDE solution formulations
3. Probability density of solution
4. Moments of solution

Important notions I

$(\Omega, \mathbb{F}, \mathcal{P})$ well-defined probability space.
$\boldsymbol{x}(t)$: stochastic process.

- An Itô process solves the following SDE starting at $\boldsymbol{x}(0)$:

$$d\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}, t)dt + \boldsymbol{L}(\boldsymbol{x}, t)d\boldsymbol{\beta}$$

- The *available information* at time t about process $\boldsymbol{x}(t)$ denoted by $\{\mathcal{F}_t\} \subseteq \mathbb{F}$ is called a *filtration*.
- $\boldsymbol{x}(t)$ is called a *martingale* iff it has bounded expectation and it holds that $\mathbb{E}[\boldsymbol{x}(t)|\mathcal{F}_s] = \boldsymbol{x}(s) \;\; \forall t \geq s$.
- $\boldsymbol{x}(t)$ is a *Markov process* iff it is true that $p(\boldsymbol{x}(t)|\mathcal{F}_s) = p(\boldsymbol{x}(t)|\boldsymbol{x}(s)) \; \forall t \geq s$.

Important notions II

- The *generator* of an Itô process is a differential operator

$$\mathcal{A}_t(\bullet) = \frac{\partial(\bullet)}{\partial t} + \sum_i \frac{\partial(\bullet)}{\partial x_i} f_i(\boldsymbol{x}, t) + \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2(\bullet)}{\partial x_i \partial x_j} \right) [\boldsymbol{L}(\boldsymbol{x}, t) \boldsymbol{Q} \boldsymbol{L}^T(\boldsymbol{x}, t)]_{i,j}$$

- A SDE solution is *weak* iff we can construct $\hat{\beta}(t)$, $\hat{\boldsymbol{x}}(t)$ such that the pair is a solution to the SDE.
- A solution to the *martingale problem* (MP) for generator $\mathcal{A}$ is a Markov process $\boldsymbol{x}(t)$ for which

$$h(\boldsymbol{x}(t)) - \int_0^t \mathcal{A} h(\boldsymbol{x}(s)) ds$$

is a martingale.

Existence and uniqueness of SDE solution

$$d\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}, t)dt + \boldsymbol{L}(\boldsymbol{x}, t)d\boldsymbol{\beta}$$

- **Theorem**:

  $\boldsymbol{x}(t)$ weak solution iff $\boldsymbol{x}(t)$ solves the MP.

- **Corollary**:

  existence of weak solution $\equiv$ existence of *some* solution to MP
  uniqueness in law $\equiv$ existence of *at most one* solution to MP

- Equivalence between weak solution and MP formulations of SDE.
- Benefits of using martingale formulation: theory of weak convergence, regular conditional probabilities, localisation.

Kolmogorov's forward equation

- We know solution at time t_0 in the form of $p(\boldsymbol{x}(t_0))$.
- What about time $t \geq t_0$?
- **Forward Kolmogorov equation**: $p(\boldsymbol{x}(t))$ of the solution solves the IVP with initial condition $p(\boldsymbol{x}(t_0))$:

$$\frac{\partial p(\boldsymbol{x}(t))}{\partial t} = -\sum_i \frac{\partial}{\partial x_i} [f_i(\boldsymbol{x}, t) p(\boldsymbol{x}(t))] + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \{[\boldsymbol{L}(\boldsymbol{x}, t) \boldsymbol{Q} \boldsymbol{L}(\boldsymbol{x}, t)]_{ij} p(\boldsymbol{x}(t))\}$$

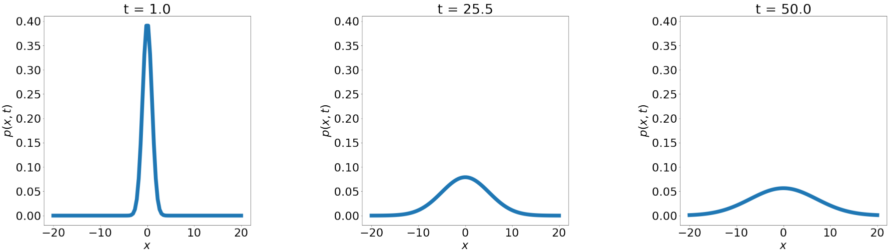
- This looks similar to applying the generator $\mathcal{A}$ to $p(\boldsymbol{x}(t))$.

$$\frac{\partial p}{\partial t} = \mathcal{A}^* p$$

with $\mathcal{A}^*$ adjoint operator of $\mathcal{A}$.

Applications of forward equation I

- **Example**: The SDE $dx = d\beta$ with constant diffusion $q = 2D$ reduces to the diffusion equation
$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}.$$
- whose solution given the initial condition $p(x(0)) = \delta(x)$ is
$$p(x(t)) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$



Result for time-independent processes

- For time-independent processes like

$$d\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})dt + \boldsymbol{L}(\boldsymbol{x})d\boldsymbol{\beta}$$

with diffusion matrix $\boldsymbol{Q} = q\boldsymbol{I}$ the forward equation satisfies

$$\frac{\partial p(\boldsymbol{x}(t))}{\partial t} = 0.$$

- If we can transform the SDE into

$$d\boldsymbol{x} = -\frac{1}{2}\nabla v(\boldsymbol{x})dt + \boldsymbol{L}(\boldsymbol{x})d\boldsymbol{\beta}$$

via $\boldsymbol{f}(\boldsymbol{x}) = -\nabla v(\boldsymbol{x})$ we can then use the following result:

- **Theorem**: The solution to the time-independent forward equation is

$$p(\boldsymbol{x}) = \frac{\exp(-v(\boldsymbol{x})/q)}{\int \exp(-v(\boldsymbol{x})/q)d\boldsymbol{x}},$$

which looks like the Boltzmann-Gibbs measure.

Applications of forward equation II

- **Example**: The Ornstein-Uhlenbeck process is

$$dx = -\lambda x dt + d\beta$$

  with $x(0) = x_0$.

- Let $v(x) = \lambda x^2$ since $-\frac{1}{2}\nabla v(x) = -\lambda x$. Therefore, the probability of the solution is

$$p(x) \propto \exp\left(-\frac{\lambda x^2}{q}\right),$$

  which resembles a Gaussian distribution with zero mean and $q/2\lambda$ variance.

- Another example of applying the forward equation is found in **"Stochastic modelling of urban structure"**.

Kolmogorov's backward equation

- We have seen how the solution probability propagates forward in time.
- How can we compute moments of the solution?
- **Theorem**: $\boldsymbol{u}(\boldsymbol{x}, t) = \mathbb{E}_x[h(\boldsymbol{x}(t))]$ solves the following initial value problem with initial condition $\boldsymbol{u}(\boldsymbol{x}, 0)$:

$$\frac{\partial \mathbb{E}_x[h(\boldsymbol{x}(t))]}{\partial t} = \mathcal{A}\, \mathbb{E}_x[h(\boldsymbol{x}(t))],$$

where

$$\mathcal{A}(\bullet) = \sum_i \frac{\partial(\bullet)}{\partial x_i} f_i(\boldsymbol{x}, t) + \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2(\bullet)}{\partial x_i \partial x_j} \right) [\boldsymbol{L}(\boldsymbol{x}, t)\boldsymbol{Q}\boldsymbol{L}^T(\boldsymbol{x}, t)]_{i,j}$$

- Great! We can now use this to compute summary statistics of the Itô process.

Moments of Itô processes

- Backward Kolmogorov equation allows us to compute mean via $h(\boldsymbol{x}(t)) = x_u$ and covariance $h(\boldsymbol{x}, t) = x_u x_v - \mathbb{E}[x_u(t)]\mathbb{E}[x_v(t)]$, respectively.
- Mean $\boldsymbol{m}$ solves
$$\frac{d\boldsymbol{m}}{dt} = \mathbb{E}[\boldsymbol{f}(\boldsymbol{x}, t)]$$
while the covariance $\boldsymbol{P}$ solves
$$\frac{d\boldsymbol{P}}{dt} = \mathbb{E}[\boldsymbol{f}(\boldsymbol{x}, t)(\boldsymbol{x} - \boldsymbol{m})^T] + \mathbb{E}[(\boldsymbol{x} - \boldsymbol{m})\boldsymbol{f}^T(\boldsymbol{x}, t)] + \mathbb{E}[\boldsymbol{L}(\boldsymbol{x}, t)\boldsymbol{Q}\boldsymbol{L}^T(\boldsymbol{x}, t)]$$
- We need access to $p(\boldsymbol{x}(t))$ via the forward Kolmogorov equation which we cannot always solve.
- Free lunch only if solution to forward equation is Gaussian.

Examples

- **Example**: Ornstein-Uhlenbeck process: $dx = -\lambda x dt + d\beta$ with $x(0) = x_0$. We have

$$\frac{dm}{dt} = \mathbb{E}[-\lambda x] = -\lambda m$$

$$\frac{dP}{dt} = 2\mathbb{E}[-\lambda(x-m)^2] + \mathbb{E}[q] = -2\lambda P + q.$$

- **Example**: $dx = \sin(x)dt + d\beta$ has

$$\frac{dm}{dt} = \mathbb{E}[\sin(x)] \approx \mathbb{E}[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots].$$

- Apart from $p(\boldsymbol{x}(t))$ we also need to compute higher order moments.
- The computation cost of the n-th moment for a d-dimensional state $\boldsymbol{x}$ is $\mathcal{O}(d^n)$ and may require us to compute expectation over infinite number of moment equations in the case of $h(x) = x^n$.

Itô processes as Markov processes

- Itô processes are Markovian and are characterised by their transition densities $p(\boldsymbol{x}(t)|\boldsymbol{x}(s))$.
- **Forward Kolmogorov equation**: $p(\boldsymbol{x}(t)|\boldsymbol{y}(s))$ solves the following PDE with $t \geq s$ and initial condition $\delta(\boldsymbol{x}(s) - \boldsymbol{y}(s))$:

$$\frac{\partial p(\boldsymbol{x}(t)|\boldsymbol{y}(s))}{\partial t} = \mathcal{A}^* \, p(\boldsymbol{x}(t)|\boldsymbol{y}(s))$$

- Same applies for the backward Kolmogorov equation.
- We can now factorise the joint distribution of the solution at arbitrary time points (where SDE can be discretised) as

$$p(\boldsymbol{x}_{t_0}, \ldots, \boldsymbol{x}_{t_T}) = p(\boldsymbol{x}_{t_0}) \prod_{k=1}^{T} p(\boldsymbol{x}(t_k)|\boldsymbol{x}(t_{k-1}))$$